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Meromorphic Functions That Share Two Values*

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In this paper, we give some uniqueness theorems for meromorphic functions that share two values. Particularly, a positive answer to a question posed by Gross is derived.

1. INTRODUCTION

Let f and g be two nonconstant meromorphic functions in the complex plane. If f and g have the same a-points with the same multiplicities, we say that f and g share the value a *CM* (see [1]). It is assumed that the reader is familiar with the basic notations and the fundamental results of Nevanlinna's theory of meromorphic functions, as found in [2]. Nevanlinna proved the following well-known theorem (see [3]).

THEOREM A. *Let f and g be two nonconstant meromorphic functions. Let a_j $(j = 1, 2, 3, 4)$ be four distinct shared values CM by f and g. Then either $f \equiv g$ or f is a linear fractional transformation of g.*

In 1995, Hong-Xun Yi proved the following (see [4])

THEOREM B. *Let f and g be two nonconstant meromorphic functions such that f and g share 0, 1, and ∞ CM, and let a $(\neq 0, 1)$ be a finite value. If*

$$N\left(r, \frac{1}{f-a}\right) \neq T(r, f) + S(r, f),$$

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then f and g satisfy exactly one of the following relations:

(i) $(f-a)(g+a-1) \equiv a(1-a)$

(ii) $f+(a-1)g \equiv a$

(iii) $f \equiv ag$.

In 1976, Gross proved (see [5])

THEOREM C. *Let f and g be nonconstant entire functions, and let $S_1 = \{1\}$, $S_2 = \{-1\}$, and $S_3 = \{a, b\}$ where a and b are arbitrary constants such that $S_i \cap S_j = \emptyset$ for $i \neq j$. Suppose that $f^{-1}(S_i) = g^{-1}(S_i)$ for $i = 1, 2, 3$ with the same multiplicities. Then f and g satisfy one of the following relations*: (i) $f = g$, (ii) $fg = 1$, *or* (iii) $(f-1)(g-1) = 4$.

In [5], Gross suggested the following open question:

QUESTION 1. *Can one find two (possibly even one) finite sets S_j $(j = 1, 2)$ such that any two entire functions f and g satisfying $f^{-1}(S_j) = g^{-1}(S_j)$ counting multiplicity for $j = 1, 2$ must be identical*?

In this paper, we will establish a uniqueness theorem for meromorphic functions that share two values and give a positive answer to Gross's questions.

2. LEMMAS

LEMMA 1. *Let f and g be nonconstant meromorphic functions and let $a \neq 0$ be a finite complex number. If $f^n + a$ and $g^n + a$ share the values $0, \infty$ CM, then for any integer $n \geq 2$*

$$T(r,f) \leq \frac{n+1}{n-1} T(r,g) + S(r,f)$$

and

$$T(r,g) \leq \frac{n+1}{n-1} T(r,f) + S(r,g).$$

Proof. By the second fundamental theorem

$$\begin{aligned} nT(r,f) &= T(r, f^n + a) + O(1) \\ &\leq N(r,f) + N\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{f^n + a}\right) + S(r,f) \\ &\leq N(r,g) + N\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{g^n + a}\right) + S(r,f) \\ &\leq (n+1)T(r,g) + T(r,f) + S(r,f). \end{aligned}$$

In the same manner, we also have

$$nT(r,g) \leq (n+1)T(r,f) + T(r,g) + S(r,g).$$

Lemma 1 is proved.

LEMMA 2. *Let F and G be two meromorphic functions, and let $\phi = F'/F - G'/G$. If z_1 is a common simple zero of F and G, then*

$$\phi(z_1) = \frac{1}{2}\left(\frac{F''(z_1)}{F'(z_1)} - \frac{G''(z_1)}{G'(z_1)}\right).$$

Proof. By the Taylor expansion of F and G at z_1,

$$\frac{F'}{F} = \frac{1}{z-z_1} + \frac{F''(z_1)}{2F'(z_1)} + O((z-z_1))$$

and

$$\frac{G'}{G} = \frac{1}{z-z_1} + \frac{G''(z_1)}{2G'(z_1)} + O((z-z_1)),$$

establishing Lemma 2.

LEMMA 3. *Let f and g be two meromorphic functions, $a \neq 0$ a complex number, and $n \geq 4$ an integer. If $f^n + a$ and $g^n + a$ share the values $0, \infty$ CM and $f^n \not\equiv g^n$, then*

$$N(r,f) \leq \frac{1}{n-1}\left\{\overline{N}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{g}\right)\right\} + S(r,f).$$

Proof. Set

$$\varphi = \frac{f'}{f(f^n+a)} - \frac{g'}{g(g^n+a)}.$$

By Lemma 1 and the well-known lemma on the logarithmic derivative, we have

$$m(r,\varphi) = S(r,f).$$

Since a zero z_1 of $f^n + a$ is a simple pole of $f'/(f(f^n+a))$ and $g'/(g(g^n+a))$ with

$$\operatorname*{Res}_{z=z_1} \frac{f'}{f(f^n+a)} = \operatorname*{Res}_{z=z_1} \frac{g'}{g(g^n+a)} = -\frac{1}{na},$$

we know that the poles of φ only occur at the zeros of f and g. It follows that

$$N(r,\varphi) \le \overline{N}\left(r,\frac{1}{f}\right) + \overline{N}\left(r,\frac{1}{g}\right)$$

and

$$T(r,\varphi) \le \overline{N}\left(r,\frac{1}{f}\right) + \overline{N}\left(r,\frac{1}{g}\right) + S(r,f).$$

We suppose that $\varphi \not\equiv 0$ and notice that ∞ is a value shared CM by f and g. Since it is easily seen that a pole of order p of f is a zero of φ with order at least $np-1$, we obtain

$$\begin{aligned} nN(r,f) - \overline{N}(r,f) &\le N\left(r,\frac{1}{\varphi}\right) \le T(r,\varphi) + O(1) \\ &\le \overline{N}\left(r,\frac{1}{f}\right) + \overline{N}\left(r,\frac{1}{g}\right) + S(r,f), \end{aligned}$$

establishing Lemma 3 if $\varphi \not\equiv 0$.

We claim $\varphi \not\equiv 0$. If $\varphi \equiv 0$, then

$$n\frac{f'}{f} - n\frac{g'}{g} + \frac{ng^{n-1}g'}{g^n+a} - \frac{nf^{n-1}f'}{f^n+a} \equiv 0,$$

and so

$$\frac{f^n(g^n+a)}{g^n(f^n+a)} \equiv c,$$

where $c \ne 0$ is a constant. We rearrange this equation to obtain

$$\frac{c}{f^n} = \frac{1}{g^n} + \frac{1-c}{a}, \tag{1}$$

and notice that $c \ne 1$ since $f^n \not\equiv g^n$. We conclude that

$$T(r,f) = T(r,g) + O(1).$$

By the second fundamental theorem and (1),

$$\begin{aligned} nT(r,f) &\le \overline{N}(r,f) + \overline{N}\left(r,\frac{1}{f}\right) + \overline{N}(r,g) + S(r,f) \\ &\le 3T(r,f) + S(r,f). \end{aligned}$$

This is incompatible with $n \geq 4$, and we conclude $\varphi \not\equiv 0$. Lemma 3 is proved.

3. THEOREMS AND THEIR PROOF

THEOREM 1. *Let f and g be two meromorphic functions such that $f^n + a$ and $g^n + a$ share $0, \infty$ CM where $a \neq 0$ is a finite complex number and $n > 5$ is an integer. Then $f^n = g^n$ or $f^n g^n = a^2$, and so $f = cg$ or $fg = d$ for some constant c and d satisfying $c^n = 1$ and $d^n = a^2$.*

Proof. Since $f^n + a$ and $g^n + a$ share the value 0 and ∞ *CM*, we have

$$\frac{f^n + a}{g^n + a} = e^h,$$

where h is an entire function.

Set

$$\delta = 2h' + \left(\frac{g''}{g'} - \frac{f''}{f'}\right) + (n-1)\left(\frac{g'}{g} - \frac{f'}{f}\right).$$

Then from Lemma 1 and the well-known lemma of logarithmic derivative, we have

$$m(r, \delta) = S(r, f)$$

and

$$N(r, \delta) \leq \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{g}\right) + N_0\left(r, \frac{1}{f'}\right) + N_0\left(r, \frac{1}{g'}\right),$$

where $N_0(r, 1/f')$ and $N_0(r, 1/g')$ denote the counting functions of zeros of f' and g' which are not the zeros of $f(f^n + a)$ and $g(g^n + a)$, respectively.

We get

$$T(r, \delta) \leq \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{g}\right) + N_0\left(r, \frac{1}{f'}\right) + N_0\left(r, \frac{1}{g'}\right) + S(r, f). \tag{2}$$

We claim $\delta \equiv 0$. Suppose the contrary and consider a simple zero z_1 of $f^n + a$. By Lemma 2,

$$h'(z_1) = \left\{ \frac{(f^n + a)'}{f^n + a} - \frac{(g^n + a)'}{g^n + a} \right\}_{z=z_1}$$
$$= \frac{1}{2}\left\{(n-1)\frac{f'}{f} + \frac{f''}{f'}\right\}_{z=z_1} - \frac{1}{2}\left\{(n-1)\frac{g'}{g} + \frac{g''}{g'}\right\}_{z=z_1},$$

implying $\delta(z_1) = 0$. Thus

$$N_{1)}\left(r, \frac{1}{f^n + a}\right) \le N\left(r, \frac{1}{\delta}\right) \le T(r, \delta) + S(r, f), \tag{3}$$

where $N_{1)}(r, 1/(f^n + a))$ is the counting function of the simple zeros of $f^n + a$. By the second fundamental theorem, we have

$$\begin{aligned}
nT(r,f) &+ nT(r,g) \\
&= T(r, f^n + a) + T(r, g^n + a) + O(1) \\
&\le \overline{N}(r,f) + \overline{N}\left(r, \frac{1}{f^n + a}\right) + \overline{N}\left(r, \frac{1}{f}\right) - N_0\left(r, \frac{1}{f'}\right) + S(r,f) \\
&\quad + \overline{N}(r,g) + \overline{N}\left(r, \frac{1}{g^n + a}\right) + \overline{N}\left(r, \frac{1}{g}\right) \\
&\quad - N_0\left(r, \frac{1}{g'}\right) + S(r,g) \\
&= 2\overline{N}(r,f) + 2\overline{N}\left(r, \frac{1}{f^n + a}\right) + \overline{N}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{g}\right) \\
&\quad - N_0\left(r, \frac{1}{f'}\right) - N_0\left(r, \frac{1}{g'}\right) + S(r,f) + S(r,g).
\end{aligned}$$

Notice that

$$2\overline{N}\left(r, \frac{1}{f^n + a}\right) \le N_{1)}\left(r, \frac{1}{f^n + a}\right) + N\left(r, \frac{1}{f^n + a}\right).$$

We have from (2), (3), and Lemma 1 that

$$\begin{aligned}
n\{T(r,f) + T(r,g)\} \le \left(2 + \frac{2}{n-1}\right)&\left\{\overline{N}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{g}\right)\right\} \\
&+ N\left(r, \frac{1}{g^n + a}\right) + S(r,f),
\end{aligned}$$

and thus

$$nT(r,f) \leq \left(2+\frac{2}{n-1}\right)\left\{N\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{g}\right)\right\}+S(r,f)$$

and

$$nT(r,g) \leq \left(2+\frac{2}{n-1}\right)\left\{N\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{g}\right)\right\}+S(r,f).$$

Adding, we obtain

$$n\{T(r,f)+T(r,g)\} \leq \left(4+\frac{4}{n-1}\right)(T(r,f)+T(r,g))+S(r,f).$$

Since $n>5$, this is a contradiction, establishing $\delta\equiv 0$. Thus

$$\frac{g^{n-1}g'}{f^{n-1}f'}=ce^{-2h},$$

and hence

$$\frac{g^{n-1}g'}{(g^n+a)^2}=c\frac{f^{n-1}f'}{(f^n+a)^2}$$

and

$$\frac{1}{g^n+a}=c\frac{1}{f^n+a}+c_1,$$

where c and c_1 are constants. Consequently,

$$T(r,f)=T(r,g)+S(r,f).$$

If $c_1=0$, then $c\neq 0$ and $f^n+a=cg^n+ca$ (*i.e.*, $f^n=cg^n+a(c-1)$). If $c\neq 1$, by the second fundamental theorem

$$\begin{aligned}nT(r,f) &\leq \overline{N}(r,f)+\overline{N}\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{f^n-a(c-1)}\right)+S(r,f)\\ &=\overline{N}(r,f)+\overline{N}\left(r,\frac{1}{f}\right)+\overline{N}\left(r,\frac{1}{g^n}\right)+S(r,f)\\ &\leq 2T(r,f)+T(r,g)+S(r,f).\end{aligned}$$

In view of Lemma 1, this is a contradiction for $n>5$; hence $c=1$ and we get $f^n\equiv g^n$.

If $c_1 \neq 0$, then $c_1 f^n = e^h - c - c_1 a$. By the second fundamental theorem applied as above, we have $c + c_1 a = 0$. Also $g^n = 1/c_1 - a + ae^{-h}$. By the same reasoning, $1/c_1 = a$, and we get $f^n g^n = a^2$. Theorem 1 is thus proved.

COROLLARY. *Let f and g be two nonconstant entire functions, $a \neq 0$ be a finite complex number, and $S = \{\omega \mid \omega^n + a = 0\}$ be a set of $n(> 4)$ elements. If $f^{-1}(S) = g^{-1}(S)$ with the same multiplicities, then $f^n = g^n$ or $f^n g^n = a^2$, and so $f \equiv cg$ or $fg = d$ for some constants c and d.*

Proof. Since $N(r, f) = N(r, g) = 0$, the corollary follows from the proof of Theorem 1.

From the above results, we immediately obtain the following theorem which answers Question 1 posed by Gross.

THEOREM 2. *Let f and g be two nonconstant entire functions, $n > 4$ be an integer, and $a, b(ab \neq 0, a^{2n+2} \neq b^{2n})$ be finite complex numbers. Set*

$$S_1 = \{\omega \mid \omega^n + a = 0\} \qquad S_2 = \{\omega \mid \omega^{n+1} + b = 0\}.$$

If $f^{-1}(S_i) = g^{-1}(S_i)$ for $i = 1, 2$ with the same multiplicities, then $f \equiv g$.

Proof. By $f^{-1}(S_1) = g^{-1}(S_1)$ and the corollary of Theorem 1, we have

$$f^n = g^n \qquad \text{or} \qquad f^n g^n = a^2. \tag{4}$$

Similarly by $f^{-1}(S_2) = g^{-1}(s_2)$, we get

$$f^{n+1} = g^{n+1} \qquad \text{or} \qquad f^{n+1} g^{n+1} = b^2. \tag{5}$$

From (4) and (5), we discuss the following four cases.

(i) If $f^n = g^n$ and $f^{n+1} = g^{n+1}$, it is easily seen that $f \equiv g$.

(ii) The equations $f^n = g^n$ and $f^{n+1} g^{n+1} = b^2$ clearly cannot hold simultaneously for any sequence z_n such that $|f(z_n)| \to \infty$.

(iii) The equations $f^n g^n = a^2$ and $f^{n+1} = g^{n+1}$ clearly cannot hold simultaneously for any sequence z_n such that $|f(z_n)| \to \infty$.

(iv) If $f^n g^n = a^2$ and $f^{n+1} g^{n+1} = b^2$, we have $a^{2n+2} = b^{2n}$ which contradicts the condition of Theorem 2.

Combining (i)–(iv), Theorem 2 is proved.

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